

# Trace anomaly in the field-antifield formalism

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## Abstract

The field-antifield quantization method is used to calculate the trace anomaly for a massless scalar field in a curved background, by means of the zeta function regularization procedure.

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**1.** The quantization method of Batalin and Vilkovisky (BV) [1, 2, 3, 4] is considered to be the most powerful quantization method for theories with first-class constraints [5]. It is a Lagrangian procedure and, consequently, it is naturally covariant. The quantum action itself is the generator of the BRST transformations, in an extended space that involves the fields and the associated antifields (with opposite statistics). One of the main advantages of the BV method, besides the natural covariance, is that it can be used for gauge theories whose generators form an open algebra [2]. In this case, the *ghost-for-ghost* structure, characteristic of reducible systems [3], is build up in a very simple way.

In addition, recent results [4] show that the application of the BV formalism to anomalous gauge theories lead to a very systematic and elegant mechanism of calculating the anomalies associated to the non trivial behavior of the path integral measure and also of generating the Wess Zumino terms that compensate this contributions, leading to a BRST invariant theory. In other words, the BV method realizes, in a very natural procedure, the mechanism proposed by Faddeev and Shatashvili of restoring gauge invariance, in the BRST language.

The purpose of the present work refers just to this last case. We use the BV method to obtain the trace anomaly for a massless scalar field moving in a curved background [6]. It is important to emphasize that when the BV method is used in the computation of anomalies it must be accompanied with some regularization procedure [7]. Usually one uses the Pauli-Villars regularization method when dealing with BV quantization. We will use here the zeta function method, that can be naturally formulated in curved space.

**2.** The BV quantization procedure is defined in an enlarged space of fields and antifields, collectively denoted by  $\Phi^a$  and  $\Phi^{*a}$  respectively. The quantum action has the general  $\hbar$  expansion :

$$W(\Phi^a, \Phi^{*a}) = S(\Phi^a, \Phi^{*a}) + \sum_{p=1}^{\infty} \hbar^p M_p(\Phi^a, \Phi^{*a}). \quad (1)$$

The zero order term of the action  $W : S(\Phi^a, \Phi^{*a})$  is usually called gauge fixed action and is subject to the boundary condition:

$$S(\Phi^a, \Phi^{*a} = 0) = \mathcal{S}(\phi^i), \quad (2)$$

where  $\mathcal{S}$  is the action of the classical theory. The first order term  $M_1$  corresponds

to one loop quantum corrections. It has recently been shown that it is possible to enlarge the space of fields and antifields in such a way that  $M_1$  plays the role of Wess Zumino term, restoring the BRST invariance of the theory[4, 8]. The condition of BRST invariance of the quantum theory is translated in the BV formalism to the condition that the so called master equation must be satisfied:

$$\frac{1}{2}(W, W) = i\hbar\Delta W \quad (3)$$

where the antibracket of two generic functions  $X$  and  $Y$  is defined as:

$$(X, Y) = \frac{\partial_r X}{\partial \Phi^a} \frac{\partial_l Y}{\partial \Phi^{*a}} - \frac{\partial_r X}{\partial \Phi^{*a}} \frac{\partial_l Y}{\partial \Phi^a} \quad (4)$$

and the operator  $\Delta$  as:

$$\Delta \equiv \frac{\partial_r}{\partial \Phi^a} \frac{\partial_l}{\partial \Phi_a^*} \quad (5)$$

Anomalies correspond just to a violation in the master equation that, after a suitable regularization procedure is applied, can be written in the form [7]:

$$\frac{1}{2}(W, W) - i\hbar\Delta W = \int d^4x C^\gamma A_\gamma. \quad (6)$$

The symmetries associated to the ghosts  $C^\gamma$  are said to be broken at the quantum level. More details about the field-antifield quantization method can be found in recent reviews available in the literature [3, 4].

**3.** Let us consider a classical action representing a scalar field in a curved spacetime

$$\mathcal{S} = \frac{1}{2} \int d^4x \sqrt{-g} \left( g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi - \frac{1}{6} R \phi^2 \right), \quad (7)$$

where  $g_{\mu\nu}$  is the metric tensor and  $R$  is the curvature scalar. The Lagrangian above is invariant under the Weyl (conformal) transformations

$$\begin{aligned} \delta g_{\mu\nu}(x) &= 2\alpha(x) g_{\mu\nu}, \\ \delta \phi(x) &= -\alpha(x) \phi(x), \end{aligned} \quad (8)$$

that leads to a vanishing trace for the energy-momentum tensor. Here,  $\alpha(x)$  is some (infinitesimal) continuous and real function. It is well-known that this symmetry is not preserved in the quantum scenario [6, 9]. This corresponds to the so-called Weyl anomaly, where the trace of the energy momentum tensor is not zero anymore.

Our purpose is to calculate the anomaly, and not to find a BRST invariant representation for the theory, therefore we will consider that all the  $M_p$  terms of the quantum action  $W$  (eq. (1)) can be taken as zero. In this case, equation (6) becomes

$$-i\hbar\Delta S = \int d^4x \sqrt{-g} C A, \quad (9)$$

where the  $\sqrt{-g}$  factor was included in the volume integral since we are dealing with non flat space. First we build up the gauge fixed action  $S$  following the standard procedure [1, 3] getting:

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi - \frac{1}{12} R \phi^2 - \phi^* C \phi \right) , \quad (10)$$

where  $C$  is the ghost field associated to the symmetry (8) and, as usual, antifields are denoted by a star superscript. It is important to remark that the metric  $g_{\mu\nu}$  is taken as an external background field ( not as a quantum field) thus there is no antifield associated to it in  $S$ . Now we must calculate:

$$\Delta S = \frac{\delta^R}{\delta\phi(x)} \frac{\delta^L}{\delta\phi^*(x)} S . \quad (11)$$

As it is discussed in [7], this expression is not well defined as it would be proportional to a  $\delta(0)$ . One must regularize the action  $S$  before acting the  $\Delta$  operator on it. We will consider the zeta function regularization procedure [10].

Let us consider a complete and orthonormal set of eigenfunctions  $\phi_n$  of the operator that appears in the action (7), that is to say

$$\left( \square^2 + \frac{1}{6} \right) \phi_n = \lambda_n \phi_n , \quad (12)$$

with

$$\int d^4x \sqrt{-g} \phi_n(x) \phi_m(x) = \delta_{nm} , \quad (13)$$

$$\sum_m \phi_m(x) \phi_m(y) = \frac{\delta(x-y)}{\sqrt{-g}} . \quad (14)$$

We introduce this regularization in the last term of  $S$ , by writing it as:

$$\begin{aligned} \int d^4x \sqrt{-g} \phi^*(x) C(x) \phi(x) &= \int d^4y d^4x \sqrt{-g} \phi^*(x) C(x) \delta^4(x-y) \phi(y) \\ &= \int d^4y d^4x \sqrt{-g} \phi^*(x) C(x) \sum_m \phi_m(x) \phi_m(y) \sqrt{-g} \phi(y) . \end{aligned} \quad (15)$$

We get thus:

$$\Delta S = - \int d^4x \sqrt{-g} C \sum_m \phi_m(x) \phi_m(x) . \quad (16)$$

The previously mentioned illness in  $\Delta S$  is now contained in the sum  $\sum_m \phi_m(x) \phi_m(x)$  above. We regularize it by using the zeta function technique in curved spacetime [6, 10]. The generalized zeta function associated with the operator of eq. (2.5) is

$$\zeta(s, x) = \sum_m \frac{\phi_m(x) \phi_m(x)}{\lambda_m^s} \quad (17)$$

We thus have that the sum in equation (16) can be written in terms of the zeta function as

$$\sum_m \phi_m(x) \phi_m(x) = \lim_{s \rightarrow 0} \zeta(s, x) . \quad (18)$$

To evaluate the zeta function  $\zeta(s, x)$  at  $s = 0$  we use the heat equation associated with the operator of eq. (13), that is

$$\frac{d}{d\tau} K(x, y, \tau) + \left( \square^2 + \frac{1}{6} \right) K(x, y, \tau) = 0 , \quad (19)$$

with the condition

$$K(x, y, 0) = \frac{\delta^{(4)}(x - y)}{\sqrt{-g}} , \quad (20)$$

where  $\tau$  is a proper time evolution parameter. It is easy to see that the function  $K(x, y, \tau)$  has the form

$$K(x, y, \tau) = \sum_m e^{-\lambda_m \tau} \phi_m(x) \phi_m(y) . \quad (21)$$

It is also straightforward to express the generalized zeta function in terms of the heat kernel,

$$\begin{aligned} \zeta(x, s) &= \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} K(x, x, \tau) \\ &= \frac{1}{16\pi^2 \Gamma(s)} \int_0^\infty d\tau \tau^{s-3} \Omega(x, x, \tau) , \end{aligned} \quad (22)$$

where

$$K(x, x, \tau) = \frac{1}{(4\pi\tau)^2} \Omega(x, x, \tau) . \quad (23)$$

It is assumed that  $\Omega(x, x, \tau)$  and their derivatives are damped for large  $\tau$ . Thus, from (22) we obtain

$$\zeta(x, 0) = \frac{1}{32\pi^2} \frac{\partial^2 \Omega(x, x, \tau)}{\partial \tau^2} \Big|_{\tau=0} \quad (24)$$

For small  $x$ , one also assumes the following expansion for  $\Omega(x, x, \tau)$

$$\Omega(x, x, \tau) \sim a_0(x) + a_1(x) \tau + a_2(x) \tau^2 + 0(\tau^3) \quad (25)$$

where the coefficients  $a_m(x)$  are dimensionally independent and the first few of them have been calculated for some kind of Hermitian operators using the method of coincidence limit [11]. The combination of (18), (24) and (25) gives

$$\Delta S = - \int d^4x \sqrt{-g} C \frac{a_2}{16\pi^2} \quad (26)$$

equation (9) then tells us that the anomaly is

$$A = i\hbar \frac{a_2}{16\pi^2}, \quad (27)$$

The coefficient function  $a_2(x)$  for the operator (12) is calculated in a straightforward way [11]. The result is

$$a_2 = \frac{1}{180} \left( R_{\mu\nu\xi\rho} R^{\mu\nu\xi\rho} - R_{\mu\nu} R^{\mu\nu} - \square R \right) \quad (28)$$

This result is in agreement with previous calculations [6, 9]. Concluding, we have shown, through a typical example, how one can use the field anti-field formalism in order to calculate the trace anomaly. Our calculations provide an example of the use of an alternative regularization technique ( the zeta function ) in the BV formalism, in contrast with the usual Pauli Villars regularization found in the literature [4].

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